Assignment 3 — Solutions [*Revision* : 1.1]

Question 1

(i). To find the mass range over which helium flashes occur, I first examined the movies on the website. The $1\,M_\odot$ and $2\,M_\odot$ models show the unmistakable signature of an He flash — a sudden jump downward in luminosity in the H-R diagram, caused when the core eventually becomes non-degenerate and rapidly expands, cooling off the H-burning shell. In contrast, the $0.5\,M_\odot$ model never becomes hot enough in its core to ignite helium (thus, it cools off as a helium white dwarf), while the $5.0\,M_\odot$ model ignites helium smoothly without any luminosity jump.

(ii). Based on these initial observations, I used bisection (calculating new *EZ-Web* models midway between existing models) to narrow down the low- and high-mass boundaries of the helium flash. The results of this procedure are that the flash occurs over the interval $0.70\,M_\odot \leq M \leq 2.00\,M_\odot$, to the nearest $0.05\,M_\odot$. Fig. 1 plots the luminosity as a function of *EZ-Web* model number for the models in the vicinity of these boundaries. Observe how the $0.70\,M_\odot$ and $2.00\,M_\odot$ models show flash-originated luminosity jumps (marked with arrows), while the $0.65\,M_\odot$ and $2.05\,M_\odot$ models do not.

(iii). I calculated five *EZ-Web* models, uniformly sampling the $(0.7\,M_\odot, 2.00\,M_\odot)$ mass interval. For each model, I found the helium flash by looking for the characteristic luminosity jump (as in Fig. 1). Then, I found the helium core boundary in the corresponding structure file by searching for the innermost grid point where X (the hydrogen mass fraction) is non-negligible, and measured the mass coordinate M_r at this boundary.

The measured helium core masses are listed in Table 1. The variation in core mass, over the full mass interval considered, is at the $\sim 5\%$ level, confirming that all stars undergoing a helium flash do so with essentially the same core mass.

M ($M_\odot$)	$M_{\rm core}$ $M_\odot$
0.700	0.434
1.025	0.437
1.350	0.442
1.675	0.452
2.000	0.450

Table 1: Core masses at the helium flash for the five different models considered.

Question 2

***Stellar Interiors* Q3.2**

(i). For a 1-D random walk comprising N steps, the expectation value of the distance moved from the origin (in either direction) is $\sqrt{N}$ times the step size. Thus, for a photon to travel from the stellar center to the surface,

$$\sqrt{N}\lambda_{\rm phot} = R. \tag{1}$$

The *total* distance traveled by the photon is $L = N\lambda_{\rm phot}$; eliminating N between these two expressions gives

$$L = \left(\frac{R}{\lambda_{\rm phot}}\right)^2 \lambda_{\rm phot} = \frac{R^2}{\lambda_{\rm phot}}, \tag{2}$$

which is the desired result.

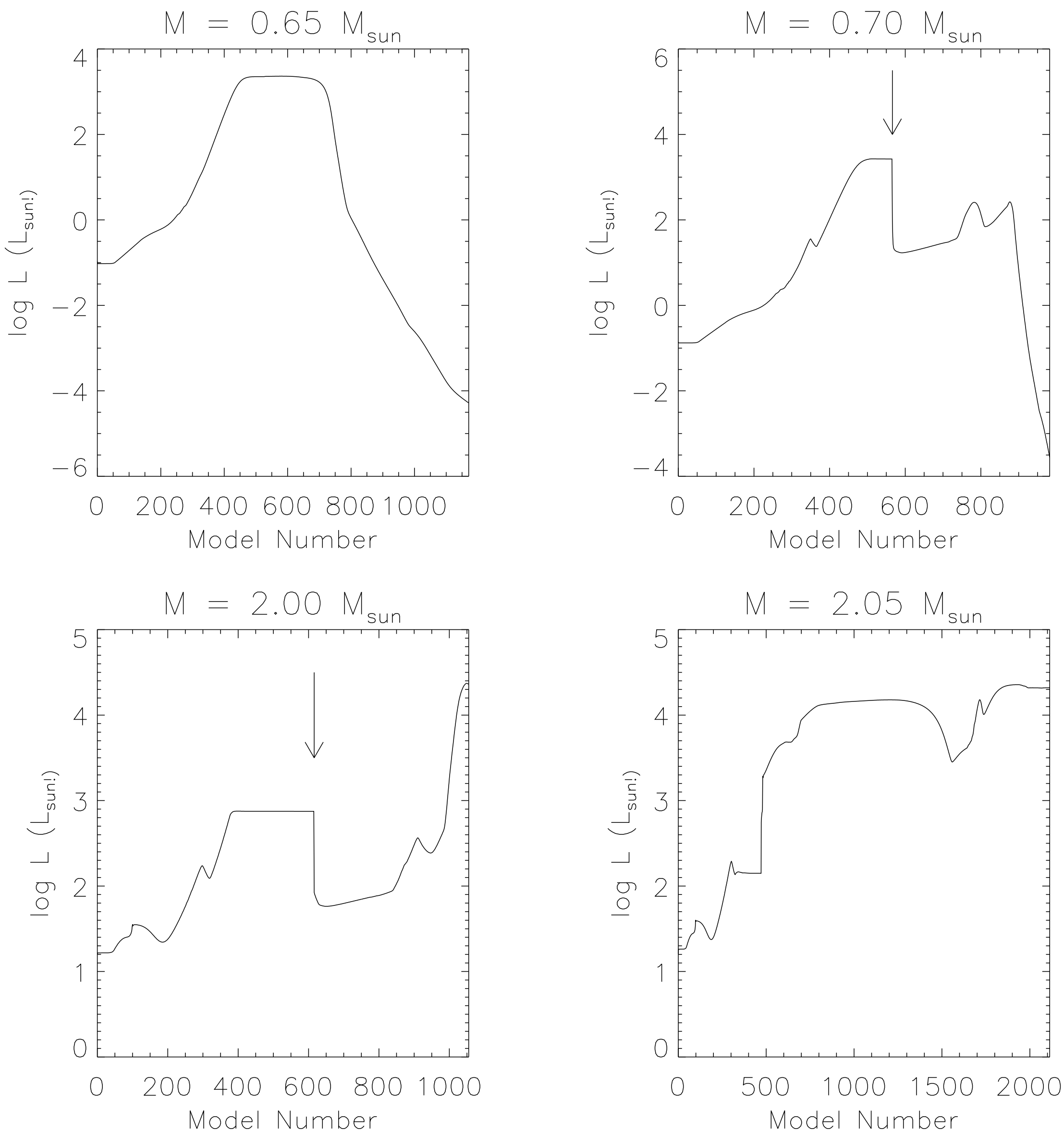

Figure 1: The luminosity as a function of model number, for models near the helium-flash boundary. The arrows indicate helium-flash episodes.

(ii). The travel time is trivially found as

$$\tau_{\rm phot} = \frac{L}{c} = \frac{R}{c}\frac{R}{\lambda_{\rm phot}}. \tag{3}$$

The first factor in the right-most expression is the direct light travel time from the stellar center to the surface, and the second factor is the correction to account for the non-direct, random-walk nature of the photon propagation.

(iii). The photon mean free path associated with electron scattering is

$$\lambda_{\rm phot} = (n_{\rm e}\sigma_{\rm e})^{-1} \tag{4}$$

Assuming a constant-density star comprised of ionized hydrogen,

$$n_{\rm e} = \frac{\rho}{u} \sim \frac{M}{R^3 u}, \tag{5}$$

where u is the atomic mass unit. Hence,

$$\lambda_{\rm phot} \sim \frac{R^3 u}{M\sigma_{\rm e}} \tag{6}$$

Combining this with eqn. (3), the travel time scales as

$$\tau_{\rm phot} \sim \frac{M\sigma_{\rm e}}{cRu} \sim 13,000 (M/M_\odot)(R/R_\odot)^{-1}\,{\rm yr}. \tag{7}$$

which is the desired result.

***Stellar Interiors* Q3.4**

Equation (3.13) of *Stellar Interiors* gives the gas pressure in terms of the momentum number distribution function $n(p)$:

$$P = \frac{1}{3}\int_0^\infty n(p)\, v\, 4\pi p^3\, {\rm d}p \tag{8}$$

The velocity is given by eqn (3.12),

$$v = \frac{\partial E}{\partial p}, \tag{9}$$

so that

$$P = \frac{1}{3}\int_0^\infty n(p)\,\frac{\partial E}{\partial p}\, 4\pi p^3\, {\rm d}p \tag{10}$$

This is completely general; however, we now assume the classical limit $\mu/kT \ll -1$, which means that the distribution function takes the form

$$n(p) \propto \exp(-E(p)/kT), \tag{11}$$

regardless of whether we are dealing with bosons or fermions. Differentiating this expression with respect to p gives

$$\frac{\partial n}{\partial p} = -\frac{n(p)}{kT}\frac{\partial E}{\partial p}; \tag{12}$$

this can be used to eliminate the energy derivative in the pressure integral (10), so that

$$P = -\frac{kT}{3}\int_0^\infty \frac{\partial n}{\partial p} 4\pi p^3\, {\rm d}p. \tag{13}$$

Integrating by parts,

$$P = -\frac{kT}{3}\left\{\left[4\pi p^3 n(p)\right]_0^\infty - 3\int_0^\infty 4\pi p^2 n(p)\,\mathrm{d}p\right\} \tag{14}$$

The first term on the right-hand side (in square brackets) must vanish, because $n(p)$ is finite as $p \to 0$, and likewise decays exponentially as $p \to \infty$. The integral in the second term can be recognized as the total number density n (cf. *Stellar Interiors*, eqn. 3.10), so that

$$P = kTn = nkT, \tag{15}$$

irrespective of whether the particles are relativistic, non-relativistic or in-between. This is the desired result.

***Stellar Interiors* Q3.6**

(i). Eqn. (3.27) of *Stellar Interiors* indicates that, for an ideal (classical) gas,

$$P \propto \exp(\mu_{\mathrm{local}}/kT). \tag{16}$$

Thus,

$$\left(\frac{\partial P}{\partial \mu_{\mathrm{local}}}\right)_T = \frac{P}{kT}. \tag{17}$$

With the ideal gas law $P = \rho kT/m$, this corresponds to

$$\left(\frac{\partial P}{\partial \mu_{\mathrm{local}}}\right)_T = \frac{\rho}{m}. \tag{18}$$

(ii). It transpires that there are a couple of errors in the introductory text of the question, that make this part impossible to answer as-is. There is a sign error on the potential ψ (it should get more *positive* with increasing height); and the total chemical potential is only constant for an *isothermal* atmosphere (see, e.g., Sturge 2003, *Statistical and Thermal Physics: Fundamentals and Applications*, Q9.6)

So, let's flip the sign of ψ and presume an isothermal atmosphere. Then, because $\mu_{\mathrm{tot}} \equiv \mu_{\mathrm{local}} + \phi$ is a constant, we have

$$\frac{\mathrm{d}\mu_{\mathrm{tot}}}{\mathrm{d}z} = \frac{\mathrm{d}\mu_{\mathrm{local}}}{\mathrm{d}z} + \frac{\mathrm{d}\phi}{\mathrm{d}z} = 0, \tag{19}$$

whence

$$\frac{\mathrm{d}\mu_{\mathrm{local}}}{\mathrm{d}z} = -\frac{\mathrm{d}\phi}{\mathrm{d}z} = -mg. \tag{20}$$

Via the chain rule, the pressure gradient is then given by

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \left(\frac{\partial P}{\partial \mu_{\mathrm{local}}}\right)_T \frac{\mathrm{d}\mu_{\mathrm{local}}}{\mathrm{d}z} = -g\rho, \tag{21}$$

which is the desired result.