(Mostly) Radiative Stars

In *Handout* xix we showed that fully convective stars lie on almost vertical lines[1] in the Hertzsprung-Russell diagram. What about the opposite case, of fully radiative stars? Let's start by recalling the scaling relations for the central pressure and central temperature of stars, assuming an ideal-gas equation of state:

$$P_{\rm c} \sim \frac{M^2}{R^4}, \qquad T_{\rm c} \sim \frac{M}{R} \tag{1}$$

(see *Handout* xix). If we assume that energy is transported by radiation, then the radiative diffusion equation (eqn. 6 of *Handout* xii) gives an additional scaling relation for the luminosity

$$L \sim \frac{R^4 T_{\rm c}^4}{\kappa M}. \tag{2}$$

For simplicity, we'll assume that the opacity κ doesn't depend on temperature or density, and therefore drop it from subsequent expressions. Then, combining these three scaling relations, we arrive at the *mass-luminosity relation for (mostly) radiative stars:*

$$L \sim M^3. \tag{3}$$

This result tells us that radiative stars lie on horizontal (constant-L) lines in the HR diagram. We've encountered these horizontal lines before: they are Henyey tracks (see Fig. 1).

[1] Hayashi tracks, of course!

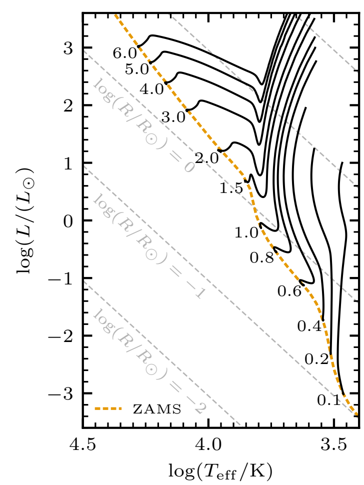

Figure 1: Evolutionary tracks in the Hertzsprung-Russell diagram for stars in the pre-main sequence phase, calculated using *MESA*. Each track is labeled at the zero-age main sequence (ZAMS) with its initial mass in $M_\odot$. With increasing mass, the horizontal Henyey tracks are displaced toward higher luminosities — exactly as predicted by eqn. (3). Taken from Fig. 1 of *Handout* viii).

Main-Sequence Stars

As well as describing Henyey tracks, the mass-luminosity relation (3) does a pretty good job of reproducing the behavior of main-sequence stars[2] . Fig. 2 shows that the exponent is closer to 3.5 than 3 (due largely to the dependence of opacity on temperature and density, which we neglected in our analysis); but we're not off by much.

With a little further effort, we can derive a similar scaling relation for the radius of main sequence stars. We begin by writing

$$L = L_{\rm nuc} \sim \epsilon_{\rm nuc} M, \tag{4}$$

where the first equality here follows from the definition of the main sequence[3]. Let's express the nuclear energy generation rate per unit mass $\epsilon_{\rm nuc}$ in the form

$$\epsilon_{\rm nuc} = \epsilon_{\rm nuc,0}\, \rho^\alpha T^\beta, \tag{5}$$

where $\epsilon_{\rm nuc,0}$, α and β are constants. Then, combining this with eqns. (1,3,4) and $\rho \sim M/R^3$, and then solving for the radius, we obtain

$$R \sim M^{\frac{\alpha+\beta-3}{3\alpha+\beta}} \tag{6}$$

[2] This is because main sequence stars with masses $M \gtrsim 0.4\,M_\odot$ are (mostly) radiative.

[3] During the lifetime of a main-sequence star, the rate of energy generation by nuclear reactions in the core ($L_{\rm nuc}$)) precisely matches the rate at which energy is radiated into space (L).

For hydrogen burning, $\alpha = 1$ while β varies between about 5 (for burning via the pp chain) and 17 (for burning by the CNO cycle). Taking an intermediate value $\beta \approx 11$ leads to the *mass-radius relation for main-sequence stars*:

$$R \sim M^{0.64} \tag{7}$$

In reality the exponent is close to 0.8 (see Fig. 2), so again our very simple analysis isn't off by too much.

We should at this point recognize an important difference between the mass-luminosity relation (3) and the mass-radius relation (7). The former was derived with no reference to nuclear reactions, and indeed it holds for pre-main sequence stars on the Henyey track, in which there are *no* nuclear reactions. By contrast, the latter depends sensitively on which nuclear reactions are taking place, as evidenced by the exponents α and β appearing in eqn. (6).

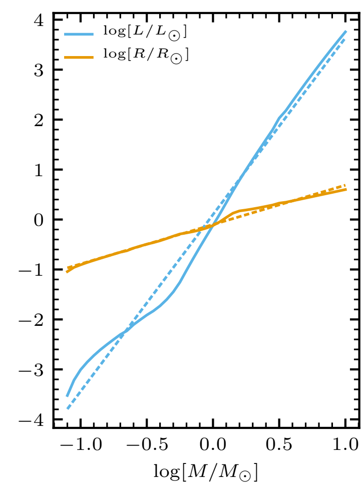

Figure 2: The logarithm of the luminosity L and radius R, plotted as a function of the logarithm of the mass M for *MESA* models of ZAMS stars. The dotted lines are linear best fits to the data, and have slopes $\mathrm{d}\log L/\mathrm{d}\log M \approx 3.5$ and $\mathrm{d}\log R/\mathrm{d}\log M \approx 0.79$.

Main-Sequence Lifetimes

Armed with the mass-luminosity relation (3), we can estimate how long stars will remain on the main sequence burning hydrogen. Suppose a star converts some fraction f of its mass from hydrogen into helium; and suppose the energy released by this conversion, per unit mass, is e. Then the total nuclear energy released during the main sequence is

$$\Delta E = f\,M\,e \tag{8}$$

This must match the luminosity radiated away during the star's main-sequence lifetime τ_{MS}:

$$\Delta E = L\,\tau_{\mathrm{MS}}. \tag{9}$$

Solving for this lifetime,

$$\tau_{\mathrm{MS}} = \frac{f\,M\,e}{L} \sim M^{-2}, \tag{10}$$

where the second equality follows from the mass-luminosity relation (3). Therefore, we see that high-mass stars have much shorter lifetimes than low-mass stars[4]. This explains an important property of HR diagrams of star clusters[5]: there exists a point on the main sequence (the *turn-off*) above which stars are not observed because they have already completed their hydrogen burning. For progressively older clusters, the turn-off appears at progressively lower luminosities; and by measuring the position of the turn-off, we can in principle determine the age of the cluster.

[4] Even though they have more hydrogen fuel to burn, they burn it so much faster that they run out sooner.

[5] Remember that the stars in a cluster are co-eval: they were all born at (approximately) the same time, and hence have the same age. Of course, this doesn't mean that they all evolve at the same rate.

Further Reading

Kippenhahn, Weigert & Weiss, §22.1; *Ostlie & Carroll*, §10.6.