Ionization of Hydrogen
Revision : 1.2

The ionization state of a gas in equilibrium at temperature T can be found using Saha's equation,

$$\frac{N_{j+1}}{N_j} = \frac{2Z_{j+1}}{n_\mathrm{e}Z_j} \left(\frac{2\pi m_\mathrm{e}kT}{h^2}\right)^{3/2} e^{-\chi_j/kT}. \tag{1}$$

There is a separate Saha equation for each pair $(j, j+1)$ of ionization states associated with each element present in the gas. These equations have to be solved simultaneously, together with an equation governing charge conservation which ultimately sets the electron number density n_e. Generally, this is a task best left to a computer.

However, the simplest case of a pure hydrogen gas *is* amenable to analytic solution, since there is only one Saha equation to solve. Assuming most of the neutral hydrogen is in the ground state, we can make the approximation $Z_\mathrm{I} \approx g_1 = 2$ (here, g_1 is the statistical weight of the ground state). For the ionized hydrogen, $Z_\mathrm{II} = 1$ because the ionized state is just a proton. Saha's equation then becomes

$$\frac{N_\mathrm{II}}{N_\mathrm{I}} = \frac{1}{n_\mathrm{e}} \left(\frac{2\pi m_\mathrm{e}kT}{h^2}\right)^{3/2} e^{-13.6\,\mathrm{eV}/kT}. \tag{2}$$

To determine the electron number density n_e, we can take advantage of charge conservation. If there are n hydrogen atoms/ions per unit volume, then

$$n_\mathrm{e} = \frac{N_\mathrm{II}}{N_\mathrm{I} + N_\mathrm{II}}n, \tag{3}$$

since there is one electron for each H II ion. Then, Saha's equation becomes

$$\frac{N_\mathrm{II}^2}{N_\mathrm{I}(N_\mathrm{I} + N_\mathrm{II})} = \frac{N_\mathrm{II}^2}{(N - N_\mathrm{II})N} = \frac{1}{n} \left(\frac{2\pi m_\mathrm{e}kT}{h^2}\right)^{3/2} e^{-13.6\,\mathrm{eV}/kT}, \tag{4}$$

where $N \equiv N_\mathrm{I} + N_\mathrm{II}$. Introducing the ionization fraction $x = N_\mathrm{II}/N$, this becomes

$$\frac{x^2}{1 - x} = \frac{1}{n} \left(\frac{2\pi m_\mathrm{e}kT}{h^2}\right)^{3/2} e^{-13.6\,\mathrm{eV}/kT}. \tag{5}$$

This last equation can be recognized as a quadratic equation for x, which can be solved using standard methods. Based on the resulting solution, Fig. 1 plots the ionization fraction as a function of temperature for a pure hydrogen gas with a number density $n = 10^{20}\,\mathrm{m}^{-3}$ (a value typical to a stellar atmosphere). Note how a fraction $x = 1/2$ is reached at a temperature around $T \approx 10,000\,\mathrm{K}$ — this is where the Balmer lines are strongest.
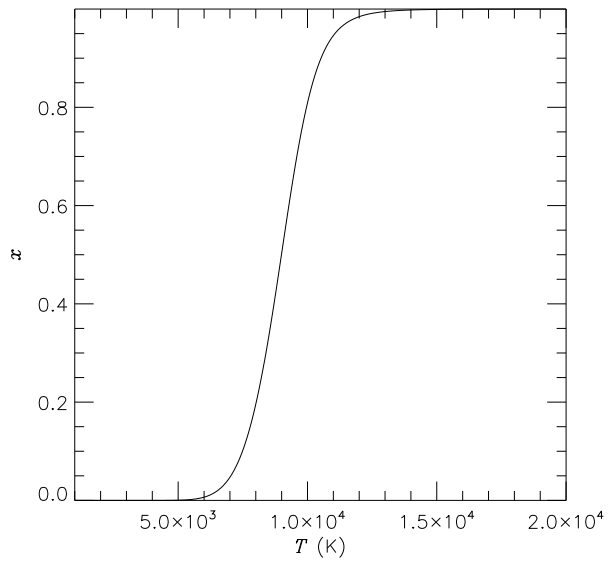
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Figure 1: The ionization fraction x of a pure hydrogen gas, plotted as a function of temperature.