12 — Atmospheric Transfer [$Revision : 1.3$]

- Plane-parallel atmospheres
  - Recall radiative transfer equation:
  $$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$
  - In this form, difficult to apply to a stellar atmosphere, because rays in different directions sample different ρ & κ_λ — optical depth scale isn't unique
  - However, for most stars (esp. main sequence), typical thickness of atmospheric layers is much smaller than stellar radius
  - Locally, atmosphere appears plane-parallel, with density, temperature, etc depending only on the vertical height z
  - Convention: $z = 0$ is top of atmosphere, z increases outwards and decreases inwards. (So, inside atmosphere, $z < 0$)
  - Define vertical optical depth:
  $$d\tau_{\lambda,v} = -\kappa_\lambda \rho dz$$
  Integrating:
  $$\tau_{\lambda,v}(z) = \int_z^0 \kappa_\lambda \rho dz'$$
  - Optical depth along non-vertical ray given by
  $$\tau_\lambda = \tau_{\lambda,v} \sec\theta$$
  where θ is angle to vertical
  - So, RTE becomes
  $$\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda,$$
  or
  $$\mu \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda,$$
  where $\mu \equiv \cos\theta$
  - Because plane-parallel atmosphere has no preferred horizontal direction, I_λ doesn't depend on azimuthal angle ϕ: axisymmetry

- Formal solution
  - As with slab, when source function is known we can integrate RTE to find **formal solution**
  - However, because atmosphere is effectively **semi-infinite**, boundary conditions are different in this case:
    * At upper boundary $\tau_{\lambda,v} = 0$, no radiation from outside:
    $$I_\lambda = 0 \qquad \text{for } \mu < 0$$
    * At lower 'boundary' $\tau_{\lambda,v} \to \infty$, radiation field remains bounded:
    $$I_\lambda e^{-\tau_{\lambda,v}} \qquad \text{finite for } \mu > 0$$

- With these boundary conditions, formal solution is

$$I_\lambda(\tau_{\lambda,\mathrm{v}}, \mu > 0) = \int_{\tau_{\lambda,\mathrm{v}}}^{\infty} S_\lambda \mathrm{e}^{(\tau_{\lambda,\mathrm{v}}-t)/\mu} \frac{\mathrm{d}t}{\mu},$$

$$I_\lambda(\tau_{\lambda,\mathrm{v}}, \mu < 0) = \int_{\tau_{\lambda,\mathrm{v}}}^{0} S_\lambda \mathrm{e}^{(\tau_{\lambda,\mathrm{v}}-t)/\mu} \frac{\mathrm{d}t}{\mu}$$

- As specific case, the **emergent specific intensity** is given by

$$I_\lambda(\tau_{\lambda,\mathrm{v}} = 0, \mu > 0) = \int_0^\infty S_\lambda \mathrm{e}^{-t/\mu} \frac{\mathrm{d}t}{\mu}$$

which is a weighted sum of the source function over all depths

- Eddington-Barbier relation
  - Formal solution presumes we know source function
  - Approximation: near top of atmosphere, expand source function using Taylor series:

$$S_\lambda(\tau_{\lambda,\mathrm{v}}) \approx S_\lambda(0) + S'_\lambda(0)\tau_{\lambda,\mathrm{v}} = a + b\tau_{\lambda,\mathrm{v}}$$

  - Substitute into formal solution, to find emergent intensity:

$$I_\lambda(\tau_{\lambda,\mathrm{v}} = 0, \mu > 0) = a + b\mu$$

  - But this is just source function at vertical optical depth μ:

$$I_\lambda(\tau_{\lambda,\mathrm{v}} = 0, \mu > 0) = S_\lambda(\tau_{\lambda,\mathrm{v}} = \mu)$$

  - Recall that optical depth along ray in direction μ is:

$$\tau_\lambda = \tau_{\lambda,\mathrm{v}} \sec\theta = \frac{\tau_{\lambda,\mathrm{v}}}{\mu}$$

    Hence:

$$I_\lambda(\tau_{\lambda,\mathrm{v}} = 0, \mu > 0) = S_\lambda(\tau_\lambda = 1)$$

  - This is the **Eddington-Barbier relation** (EBR): for a source function $S_\lambda = a + b\tau_{\lambda,\mathrm{v}}$, the emergent specific intensity along a given ray is just equal to the source function at optical depth unity along that ray
  - Not entirely accurate, but gives important **insight** into limb darkening and spectral line formation

- Limb darkening
  - Generally the case that S_λ increases with increasing τ_λ
  - For instance, with blackbody, $S_\lambda = B_\lambda$, and B_λ gets larger in the deeper, hotter layers of atmosphere
  - Applying EBR, emergent specific intensity will be larger along vertical rays ($\mu \approx 1$) than horizontal rays ($\mu \approx 0$)
  - Phenomenon of **limb darkening**; easily seen in photographs of Sun, and also shows up in eclipsing binaries

- Spectral lines
  - A **spectral line** is formed when κ_λ becomes larger over a narrow range of wavelengths
  - In this narrow range, $\tau_\lambda = 1$ level is much higher in atmosphere, where source function is lower
  - Therefore, by EBR, emergent intensity is lower $\longrightarrow$ **absorption line**