Assignment 4 — due November 14$^{\text{th}}$ [$Revision$: 1.2]

1. The first moment of the wavelength-dependent radiative transfer equation is

$$-\frac{1}{\kappa_\lambda \rho}\frac{\mathrm{d}P_{\mathrm{rad},\lambda}}{\mathrm{d}r} = \frac{F_\lambda}{c}$$

where

$$P_{\mathrm{rad},\lambda} = \frac{1}{c}\int I_\lambda \mu^2 \,\mathrm{d}\Omega$$

is the radiation pressure per unit wavelength interval, and

$$F_\lambda = \int I_\lambda \mu \,\mathrm{d}\Omega$$

is the corresponding radiative flux.

(a) Deep inside the star, the radiation pressure is given by

$$P_{\mathrm{rad},\lambda} = \frac{4\pi}{3c} B_\lambda(T)$$

where B_λ is the Planck function. Derive an expression for F_λ in these regions, in terms of the temperature gradient $\mathrm{d}T/\mathrm{d}r$ and other quantities.

(b) Integrate your answer over all wavelengths to find an exression for the bolometric radiative flux F (your answer should contain an integral involving $1/\kappa_\lambda$ and other terms).

(c) By writing your result in the form

$$F = -\frac{4acT^3}{3\bar{\kappa}\rho}\frac{\mathrm{d}T}{\mathrm{d}r},$$

confirm that the Rosseland mean opacity $\bar{\kappa}$ is defined as

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\lambda}\frac{\mathrm{d}B_\lambda}{\mathrm{d}T}\,\mathrm{d}\lambda}{\int_0^\infty \frac{\mathrm{d}B_\lambda}{\mathrm{d}T}\,\mathrm{d}\lambda}$$

2. Assume that the pressure and density structure of the Sun can be approximated by an $n = 3$ polytrope, for which

$$\xi_1 = 6.897, \qquad -\xi_1^2 \frac{\mathrm{d}D_n}{\mathrm{d}\xi}\bigg|_{\xi_1} = 2.018.$$

(a) Calculate the radius constant λ_n

(b) Calculate the central density ρ_{c}

(c) Calculate the central pressure P_{c}

(d) Calculate the central temperature T_{c}, assuming that the radiation pressure is negligible, and that the mean molecular weight $\mu = 0.62$.

3. In a white dwarf stars the density is so high that the electrons are degenerate; as a result, the gas does not obey an ideal equation of state, but instead follows

$$P = A\rho^{5/3}$$

where A is a physical constant (note the lack of any dependence on temperature). Demonstrate that these stars obey the mass-radius relation

$$R \propto M^{-1/3}.s$$

4. Use *EZ Web* to calculate a $10\,M_\odot$ ZAMS stellar model. On the submission form, set 'Maximum Model Number' to 0, and 'Create Model Files?' to 'Yes'; the resulting ZIP archive will then contain a single, ZAMS stellar model named `model_00000.txt`. From the model data, plot a line graph of $\log P$ as a function of $\log \rho$. Estimate the gradient of the curve, and from that calculate an approximate polytropic index for the model.

5. For the ZAMS model from the previous question, plot the logarithmic temperature gradient $\mathrm{d}\ln T/\mathrm{d}\ln r$ as a function of $\log T$ (we use $\log T$ as the x-axis to allow the outer layers of the star to be adequately resolved). To calculate this gradient, note that column 17 of the model file gives the 'physical temperature gradient'

$$\nabla = \frac{\mathrm{d}\log T}{\mathrm{d}\log P};$$

using the chain rule, the temperature gradient can be written

$$\frac{\mathrm{d}\log T}{\mathrm{d}\log r} = \nabla \frac{\mathrm{d}\log P}{\mathrm{d}\log r} = \frac{r\nabla}{P}\frac{\mathrm{d}P}{\mathrm{d}r},$$

where the pressure gradient is calculated from the equation of hydrostatic equilibrium.

6. Using the temperature gradient found in the preceding question, plot a curve showing the fractional radiative luminosity L_{rad}/L as a function of r/R. In the same diagram, also plot the fractoinal interior luminosity L_r/L. In what regions of the star do the two curves differ — and why?