Assignment 3 — Solutions [$Revision: 1.2$]

1. (a) The mean intensity is given by

$$\langle I \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} (I_0 + I_1 \mu) \, \mathrm{d}\mu \, \mathrm{d}\phi = I_0.$$

The flux is given by

$$F = \int_0^{2\pi} \int_{-1}^{1} (I_0 + I_1 \mu) \mu \, \mathrm{d}\mu \, \mathrm{d}\phi = \frac{4\pi}{3} I_1.$$

The radiation pressure is given by

$$P_{\mathrm{rad}} = \frac{1}{c} \int_0^{2\pi} \int_{-1}^{1} (I_0 + I_1 \mu) \mu^2 \, \mathrm{d}\mu \, \mathrm{d}\phi = \frac{4\pi}{3c} I_0.$$

[3 points]

(b) Both the mean intensity and the radiation pressure depend only on I_0; eliminating I_0 between them leads to the Eddington approximation,

$$P_{\mathrm{rad}} = \frac{4\pi}{3c} \langle I \rangle.$$

[1 points]

(c) From the above expressions for the mean intensity and flux, the solution to the RTE becomes

$$I_0 = I_1 \left(\tau_{\mathrm{v}} + \frac{2}{3} \right).$$

[2 points]

(d) In the limit $\tau_{\mathrm{v}} \gg 1$, the above expression for I_0 becomes

$$I_0 \approx I_1 \tau_{\mathrm{v}},$$

which indeed is very much greater than I_1.

[2 points]

2. (a) In radiative equilibrium for a gray atmosphere, $S = \langle I \rangle$. Using the expression for the solution of the RTE (see Q1), this in turn implies that

$$S = \frac{3}{4\pi} F \left(\tau_{\mathrm{v}} + \frac{2}{3} \right).$$

[2 points]

(b) The formal solution for upward ($\mu > 0$) radiation is

$$I(\tau_{\mathrm{v}}, \mu > 0) = \int_{\tau_{\mathrm{v}}}^{\infty} S \mathrm{e}^{(\tau_{\mathrm{v}} - t)/\mu} \frac{\mathrm{d}t}{\mu},$$

(recall that t is a dummy integration variable). Substituting in the above expression for S, we have

$$I(\tau_{\mathrm{v}}, \mu > 0) = \int_{\tau_{\mathrm{v}}}^{\infty} \frac{3}{4\pi} F \left(t + \frac{2}{3} \right) \mathrm{e}^{(\tau_{\mathrm{v}} - t)/\mu} \frac{\mathrm{d}t}{\mu} = \frac{3}{4\pi} F \left(\mu + \tau_{\mathrm{v}} + \frac{2}{3} \right).$$

[3 points]

(c) Integrating, the upward flux is

$$F_+ = \int_0^{2\pi} \int_0^1 \frac{3}{4\pi} F \left(\mu + \tau_\mathrm{v} + \frac{2}{3} \right) \mu \, \mathrm{d}\mu = F \left(1 + \frac{3}{4} \tau_\mathrm{v} \right).$$

[2 points]

(d) The downward component is given by

$$F_- = F - F_+ = -F \frac{3}{4} \tau_\mathrm{v}$$

This is negative for $\tau_\mathrm{v} > 0$, which is not physically possible. This is a consequence of the fact that the Eddington approximation produces atmosphere models that are not in radiative equilibrium.

[2 points]

3. (a) For an ideal gas,

$$\rho = \frac{P\mu}{kT}.$$

Thus, the equation of hydrostatic equilibrium becomes

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_\odot P\mu}{r^2 kT}.$$

Solving,

$$\ln P = \frac{GM_\odot \mu}{rkT} + C$$

where C is a constant of integration. This constant can be found by noting that at $r = R_\odot$, $P(r) = P(R_\odot)$; so:

$$C = \ln P(R_\odot) - \frac{GM_\odot \mu}{R_\odot kT}$$

Hence,

$$P(r) = P(R_\odot) \exp \left[\frac{GM_\odot \mu}{kT} \left(\frac{1}{r} - \frac{1}{R_\odot} \right) \right]$$

(here, $\exp[x]$ is being used as an alternative way of writing e^x).

[5 points]

(b) With $z = r - R_\odot$,

$$P(r) = P(R_\odot) \exp \left[\frac{GM_\odot \mu}{kT} \left(\frac{1}{R_\odot + z} - \frac{1}{R_\odot} \right) \right]$$

[1 points]

(c) When $z \ll R_\odot$, we can make the approximation

$$\frac{1}{R_\odot + z} \approx \frac{1}{R_\odot} - \frac{z}{R_\odot^2}$$

Hence, the pressure is

$$P(r) \approx P(R_\odot) \exp \left[\frac{GM_\odot \mu}{kT} \left(\frac{1}{R_\odot} - \frac{z}{R_\odot^2} - \frac{1}{R_\odot} \right) \right] = P(R_\odot) \exp \left[-\frac{GM_\odot \mu}{kT} \frac{z}{R_\odot^2} \right]$$

This can be written as

$$P(r) = \exp\left[-\frac{z}{H_P}\right]$$

where

$$H_P = \frac{kTR_\odot^2}{GM_\odot\mu} = \frac{kT}{g\mu}$$

is the pressure scale height at the stellar surface $r = R_\odot$. These expressions are the same as in class.

[3 points]

(d) For $z \gg R_\odot$, the asymptotic value of the pressure is

$$P(\infty) = P(R_\odot)\exp\left[-\frac{GM_\odot\mu}{kTR_\odot}\right].$$

[2 points]

(e) Inserting the values given, the asymptotic pressure is found as $P(\infty) \approx 0.13\,\mathrm{dyne\,cm^{-2}}$.

[1 points]

(f) This is much larger than the ISM pressure; hence, the solar corona will expand out into the ISM, leading to a *solar wind*.

[2 points]