3 — Blackbody Radiation [$Revision : 1.5$]

- Stellar spectra
  - Bolometric flux F measures total energy at all wavelengths
  - Much more information available from considering spectrum:

$$F_\nu \, \mathrm{d}\nu \leftrightarrow \text{Energy/unit second/unit area between frequencies } (\nu, \nu + \mathrm{d}\nu)$$

$$F_\lambda \, \mathrm{d}\lambda \leftrightarrow \text{Energy/unit second/unit area between wavelengths } (\lambda, \lambda + \mathrm{d}\lambda)$$

  - Important:

$$F_\nu \, \mathrm{d}\nu = F_\lambda \, \mathrm{d}\lambda$$

    But:

$$\nu = \frac{c}{\lambda} \longrightarrow \mathrm{d}\nu = \frac{c}{\lambda^2} \mathrm{d}\lambda$$

    Hence:

$$F_\nu = \frac{c}{\lambda^2} F_\lambda$$

  - Integrating over all frequencies/wavelengths:

$$F = \int_0^\infty F_\nu \, \mathrm{d}\nu = \int_0^\infty F_\lambda \, \mathrm{d}\lambda$$

  - In the UV, visible and IR parts of spectrum of many stars, F_λ is crudely approximated as a blackbody
- Blackbody radiation
  - **Blackbody** (BB) is an object that absorbs *all* radiation falling on it
  - In thermal equilibrium at temperature T, radiation re-emitted by BB has a unique F_λ that depends only on T
  - Good approximation to BB is opaque container with small hole in it **hohlraum**
  - General features of BB radiation:
    * Single peak described by **Wien's law**: $\lambda_{\max} T = 0.290 \, \mathrm{cmK}$
    * Steep decline blueward of peak ($\lambda < \lambda_{\max}$)
    * Shallow decline redward of peak ($\lambda > \lambda_{\max}$) — **Rayleigh-Jeans tail**
- Rayleigh-Jeans formula
  - From classical (pre-quantum) physics
  - BB radiation inside cavity of dimensions $L_x \times L_y \times L_z$ is superposition of standing waves
  - Each wave described by wavevector $\mathbf{k} = (k_x, k_y, k_z)$; $k \equiv |\mathbf{k}| = 2\pi/\lambda$
  - Boundary conditions:

$$k_x = \frac{\pi n_x}{L_x} \qquad k_y = \frac{\pi n_y}{L_y} \qquad k_z = \frac{\pi n_z}{L_z}$$

    for integer n_x, n_y, n_z
  - Number of permitted waves in interval $(k, k + \mathrm{d}k)$:

$$N_k \mathrm{d}k = \frac{4\pi k^2 \mathrm{d}k}{8} \times \frac{2 L_x L_y L_z}{\pi^3}$$

    First factor is volume of k-space *octant* between $(k, k + \mathrm{d}k)$; second factor is number of standing waves in volume of k space (extra 2 due to two possible polarization directions)

- Number of standing waves in interval $(\lambda, \lambda + d\lambda)$:

$$N_\lambda d\lambda = N_k dk = \frac{8\pi}{\lambda^4} L_x L_y L_z d\lambda$$

- Classical physics: in thermal equilibrium, an oscillator (standing wave) gets energy kT (**equipartition**). IMPORTANT: k is now Boltzmann's constant, *not* the wavevector!
- Thus, **energy density** (energy per unit volume):

$$u_\lambda d\lambda = \frac{kTN_\lambda}{L_x L_y L_z} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

- Planck formula
  - At short wavelengths, Rayleigh-Jeans u_λ blows up (**ultraviolet catastrophe**)
  - Fix is energy quantization: energy of standing wave with frequency ν constrained to be an integer multiple of $h\nu = hc/\lambda$
  - At short wavelengths, equipartition energy kT is insufficient to make up a whole quantum
  - Leads to a turnover in the energy density distribution; full derivation gives **Planck formula**

$$u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \, d\lambda$$

  - In limit $\lambda \gg hc/kT$, Planck formula $\rightarrow$ Rayleigh-Jeans formula
- Stefan-Boltzmann formula
  - For an enclosure with energy density u_λ, flux through small hole is

$$F_\lambda \, d\lambda = \frac{c}{4} u_\lambda \, d\lambda$$

    (will prove when we do radiative transfer).
  - For BB:

$$F_\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \, d\lambda$$

  - Integrate over all wavelengths gives **Stefan-Boltzmann formula**:

$$F = \int_0^\infty F_\lambda d\lambda = \sigma T^4$$

    where $\sigma = 5.670 \times 10^{-5}\, \mathrm{erg s^{-1} cm^{-2} K^{-4}}$.
  - For star of radius R

$$L = 4\pi R^2 F_{\rm surface} = 4\pi d^2 F_{\rm obs}$$

    where $F_{\rm surf}$ is bolometric surface flux, and $F_{\rm obs}$ is observed flux at distance d. Assuming star is BB,

$$F_{\rm obs} = \left(\frac{R}{d}\right)^2 F_{\rm surface} = \left(\frac{R}{d}\right)^2 \sigma T^4$$

  - In reality, stars are not BBs. However, we define **effective temperature** $T_{\rm eff}$ of a star as temperature of a BB having same bolometric surface flux:

$$T_{\rm eff} = \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4}$$

    Important: $T_{\rm eff}$ is measure of surface flux, it is *not* surface temperature (although indirectly related to surface temperature)