

DZIEM Formulation

Radial Adiabatic

Variables

$$\begin{aligned}x &= \frac{r}{R_*} \\ y_1 &= x^2 \frac{\xi_r}{r} \\ y_2 &= x^2 \frac{1}{gr} \left(\frac{p'}{\rho} \right)\end{aligned}$$

Differential Equations

$$\begin{aligned}x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 \right) y_1 + -\frac{V}{\Gamma_1} y_2 \\ x \frac{dy_2}{dx} &= (c_1 \omega^2 + U - A^*) y_1 + (3 - U + A^*) y_2\end{aligned}$$

Boundary Conditions

$$y_1 = 0 \quad \text{as } x \rightarrow 0$$

$$y_1 - y_2 = 0 \quad \text{as } x \rightarrow 1 \quad (\text{Zero})$$

Non-Radial Adiabatic

Variables

$$\begin{aligned}
 x &= \frac{r}{R_*} \\
 y_1 &= x^{2-\ell} \frac{\xi_r}{r} \\
 y_2 &= x^{2-\ell} \frac{1}{gr} \left(\frac{p'}{\rho} + \Phi' \right) \\
 y_3 &= x^{2-\ell} \frac{\Phi'}{gr} \\
 y_4 &= x^{2-\ell} \frac{1}{g} \frac{d\Phi'}{dr}
 \end{aligned}$$

Differential Equations

$$\begin{aligned}
 x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 - \ell \right) y_1 + \left(\frac{\ell(\ell+1)}{c_1 \omega^2} - \frac{V}{\Gamma_1} \right) y_2 + \frac{V}{\Gamma_1} y_3 \\
 x \frac{dy_2}{dx} &= (c_1 \omega^2 - A^*) y_1 + (3 - U + A^* - \ell) y_2 - A^* y_3 \\
 x \frac{dy_3}{dx} &= (3 - U - \ell) y_3 + y_4 \\
 x \frac{dy_4}{dx} &= A^* U y_1 + \frac{V}{\Gamma_1} U y_2 + \left[\ell(\ell+1) - \frac{V}{\Gamma_1} U \right] y_3 - (U + \ell - 2) y_4
 \end{aligned}$$

Boundary Conditions

$$\left. \begin{aligned} c_1 \omega^2 y_1 - \ell y_2 &= 0 \\ \ell y_3 - y_4 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 0$$

$$\left. \begin{aligned} y_1 - y_2 + y_3 &= 0 \\ U y_1 + (\ell + 1) y_3 + y_4 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 1 \quad (\text{Zero})$$

$$\left. \begin{aligned} \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - 4 - \omega^2 \right] \right\} y_1 - y_2 + \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - \ell - 1 \right] \right\} y_3 &= 0 \\ (\ell + 1) y_3 + y_4 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 1 \quad (\text{Dziembowski})$$

Non-Radial Non-Adiabatic

Variables

$$\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell} \frac{\xi_r}{r} \\
y_2 &= x^{2-\ell} \frac{1}{gr} \left(\frac{p'}{\rho} + \Phi' \right) \\
y_3 &= x^{2-\ell} \frac{\Phi'}{gr} \\
y_4 &= x^{2-\ell} \frac{1}{g} \frac{d\Phi'}{dr} \\
y_5 &= x^{2-\ell} \frac{\delta S}{c_p} \\
y_6 &= x^{-1-\ell} \frac{\delta L_{\text{rad}}}{L_*}
\end{aligned}$$

Differential Equations

Non-Adiabatic

$$\begin{aligned}
x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 - \ell \right) y_1 + \left(\frac{\ell(\ell+1)}{c_1 \omega^2} - \frac{V}{\Gamma_1} \right) y_2 + \frac{V}{\Gamma_1} y_3 + \delta y_5 \\
x \frac{dy_2}{dx} &= (c_1 \omega^2 - A^*) y_1 + (3 - U + A^* - \ell) y_2 - A^* y_3 + \delta y_5 \\
x \frac{dy_3}{dx} &= (3 - U - \ell) y_3 + y_4 \\
x \frac{dy_4}{dx} &= A^* U y_1 + \frac{V}{\Gamma_1} U y_2 + \left[\ell(\ell+1) - \frac{V}{\Gamma_1} U \right] y_3 - (U + \ell - 2) y_4 - \delta U y_5 \\
x \frac{dy_5}{dx} &= V \left[\nabla_{\text{ad}} (U - c_1 \omega^2) - 4(\nabla_{\text{ad}} - \nabla) + c_{\text{kap}} \right] y_1 + V \left[\frac{\ell(\ell+1)}{c_1 \omega^2} (\nabla_{\text{ad}} - \nabla) - c_{\text{kap}} \right] y_2 + \\
&\quad V c_{\text{kap}} y_3 + V \nabla_{\text{ad}} y_4 + [V \nabla (4 - \kappa_S) + 2 - \ell] y_5 - \frac{V \nabla}{c_{\text{rad}}} y_6 \\
x \frac{dy_6}{dx} &= \left[\ell(\ell+1) \left(\frac{\nabla_{\text{ad}}}{\nabla} - 1 \right) c_{\text{rad}} - V c_{\epsilon, \text{ad}} \right] y_1 + \left[V c_{\epsilon, \text{ad}} - \ell(\ell+1) c_{\text{rad}} \left(\frac{\nabla_{\text{ad}}}{\nabla} - \frac{3 + c'_{\text{rad}}}{c_1 \omega^2} \right) \right] y_2 + \\
&\quad \left[\ell(\ell+1) \frac{\nabla_{\text{ad}}}{\nabla} c_{\text{rad}} - V c_{\epsilon, \text{ad}} \right] y_3 + \left[c_{\epsilon, S} - \frac{\ell(\ell+1) c_{\text{rad}}}{\nabla V} - i \omega c_{\text{thm}} \right] y_5 - [1 + \ell] y_6
\end{aligned}$$

Boundary Conditions

$$\left. \begin{aligned} c_1 \omega^2 y_1 - \ell y_2 &= 0 \\ \ell y_3 - y_4 &= 0 \\ y_5 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 0$$

Outer

$$\left. \begin{aligned} y_1 - y_2 + y_3 &= 0 \\ U y_1 + (\ell + 1) y_3 + y_4 &= 0 \\ (2 - 4 \nabla_{\text{ad}} V) y_1 + 4 \nabla_{\text{ad}} V y_2 - 4 \nabla_{\text{ad}} V y_3 + 4 y_5 - y_6 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 1 \quad (\text{Zero})$$

$$\left. \begin{aligned} \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - 4 - \omega^2 \right] \right\} y_1 - y_2 + \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - \ell - 1 \right] \right\} y_3 &= 0 \\ (\ell+1)y_3 + y_4 &= 0 \\ (2 - 4\nabla_{\text{ad}}V)y_1 + 4\nabla_{\text{ad}}Vy_2 - 4\nabla_{\text{ad}}Vy_3 + 4y_5 - y_6 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 1 \quad (\text{Dziembowski})$$

JCD Formulation

Radial Adiabatic

Variables

$$\begin{aligned}x &= \frac{r}{R_*} \\ y_1 &= x^2 \frac{\xi_r}{r} \\ y_2 &= x^2 \frac{1}{\sigma^2 r^2} \left(\frac{p'}{\rho} \right)\end{aligned}$$

Differential Equations

$$\begin{aligned}x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 \right) y_1 + -\frac{V}{\Gamma_1} c_1 \omega^2 y_2 \\ x \frac{dy_2}{dx} &= \left(1 + \frac{U - A^*}{c_1 \omega^2} \right) y_1 + A^* y_2\end{aligned}$$

Boundary Conditions

$$y_1 = 0 \quad \text{as } x \rightarrow 0$$

$$y_1 - \frac{y_2}{c_1 \omega^2} = 0 \quad \text{as } x \rightarrow 1 \quad (\text{Zero})$$

Non-Radial Adiabatic

Variables

$$\left. \begin{aligned} x &= \frac{r}{R_*} \\ y_1 &= x^{2-\ell} \frac{\xi_r}{r} \\ y_2 &= x^{2-\ell} \frac{\ell(\ell+1)}{r^2 \sigma^2} \left(\frac{p'}{\rho} + \Phi' \right) \\ y_3 &= -x^{2-\ell} \frac{\Phi'}{gr} \\ y_4 &= -x^{2-\ell} r \frac{d}{dr} \left(\frac{\Phi'}{gr} \right) \end{aligned} \right\} \ell \neq 0$$

$$\left. \begin{aligned} x &= \frac{r}{R_*} \\ y_1 &= x^2 \frac{\xi_r}{r} \\ y_2 &= x^2 \frac{1}{r^2 \sigma^2} \left(\frac{p'}{\rho} \right) \\ y_3 &= -x^2 \frac{\Phi'}{gr} \\ y_4 &= -x^2 r \frac{d}{dr} \left(\frac{\Phi'}{gr} \right) \end{aligned} \right\} \ell = 0$$

Differential Equations

$$\left. \begin{aligned} x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 - \ell \right) y_1 + \left(1 - \frac{V}{\Gamma_1} \frac{c_1 \omega^2}{\ell(\ell+1)} \right) y_2 - \frac{V}{\Gamma_1} y_3 \\ x \frac{dy_2}{dx} &= \left(\ell(\ell+1) - \frac{\ell(\ell+1)}{c_1 \omega^2} A^* \right) y_1 + (A^* - \ell) y_2 + \frac{\ell(\ell+1)}{c_1 \omega^2} A^* y_3 \\ x \frac{dy_3}{dx} &= (2 - \ell) y_3 + y_4 \\ x \frac{dy_4}{dx} &= -A^* U y_1 + -\frac{V}{\Gamma_1} U \frac{c_1 \omega^2}{\ell(\ell+1)} y_2 + [\ell(\ell+1) + U(A^* - 2)] y_3 + (3 - 2U - \ell) y_4 \end{aligned} \right\} \ell \neq 0$$

$$\left. \begin{aligned} x \frac{dy_1}{dx} &= \left(\frac{V}{\Gamma_1} - 1 \right) y_1 + -\frac{V}{\Gamma_1} c_1 \omega^2 y_2 - \frac{V}{\Gamma_1} y_3 \\ x \frac{dy_2}{dx} &= \left(1 - \frac{A^*}{c_1 \omega^2} \right) y_1 + A^* y_2 + \frac{A^*}{c_1 \omega^2} y_3 \\ x \frac{dy_3}{dx} &= 2y_3 + y_4 \\ x \frac{dy_4}{dx} &= A^* U y_1 + \frac{V}{\Gamma_1} U c_1 \omega^2 y_2 + U(A^* - 2) y_3 + (3 - 2U) y_4 \end{aligned} \right\} \ell = 0$$

Boundary Conditions

$$\left. \begin{aligned} c_1 \omega^2 y_1 - \ell y_2 &= 0 \\ \ell y_3 - y_4 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 0$$

$$\left. \begin{aligned} y_1 - y_2 + y_3 &= 0 \\ U y_1 + (\ell + 1) y_3 + y_4 &= 0 \end{aligned} \right\} \text{as } x \rightarrow 1 \quad (\text{Zero})$$

$$\left\{ \begin{aligned} & \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - 4 - \omega^2 \right] \right\} y_1 - y_2 + \left\{ 1 + V^{-1} \left[\frac{\ell(\ell+1)}{\omega^2} - \ell - 1 \right] \right\} y_3 = 0 \\ & (\ell+1)y_3 + y_4 = 0 \end{aligned} \right\} \text{ as } x \rightarrow 1 \quad (\text{Dziembowski})$$

Non-Radial Non-Adiabatic

Variables

$$\begin{aligned}
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Non-Adiabatic

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x \frac{dy_2}{dx} &= (c_1 \omega^2 - A^*) y_1 + (3 - U + A^* - \ell) y_2 - A^* y_3 + \delta y_5 \\
x \frac{dy_3}{dx} &= (3 - U - \ell) y_3 + y_4 \\
x \frac{dy_4}{dx} &= A^* U y_1 + \frac{V}{\Gamma_1} U y_2 + \left[\ell(\ell+1) - \frac{V}{\Gamma_1} U \right] y_3 - (U + \ell - 2) y_4 - \delta U y_5 \\
x \frac{dy_5}{dx} &= V \left[\nabla_{\text{ad}} (U - c_1 \omega^2) - 4(\nabla_{\text{ad}} - \nabla) + c_{\text{kap}} \right] y_1 + V \left[\frac{\ell(\ell+1)}{c_1 \omega^2} (\nabla_{\text{ad}} - \nabla) - c_{\text{kap}} \right] y_2 + \\
&\quad V c_{\text{kap}} y_3 + V \nabla_{\text{ad}} y_4 + [V \nabla (4 - \kappa_S) + 2 - \ell] y_5 - \frac{V \nabla}{c_{\text{rad}}} y_6 \\
x \frac{dy_6}{dx} &= \left[\ell(\ell+1) \left(\frac{\nabla_{\text{ad}}}{\nabla} - 1 \right) c_{\text{rad}} - V c_{\epsilon, \text{ad}} \right] y_1 + \left[V c_{\epsilon, \text{ad}} - \ell(\ell+1) c_{\text{rad}} \left(\frac{\nabla_{\text{ad}}}{\nabla} - \frac{3 + c'_{\text{rad}}}{c_1 \omega^2} \right) \right] y_2 + \\
&\quad \left[\ell(\ell+1) \frac{\nabla_{\text{ad}}}{\nabla} c_{\text{rad}} - V c_{\epsilon, \text{ad}} \right] y_3 + \left[c_{\epsilon, S} - \frac{\ell(\ell+1) c_{\text{rad}}}{\nabla V} - i \omega c_{\text{thm}} \right] y_5 - [1 + \ell] y_6
\end{aligned}$$

Boundary Conditions

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Structure Coefficients

Mechanical

$$V = -\frac{d \ln p}{d \ln r} \quad A^* = \frac{1}{\Gamma_1} \frac{d \ln p}{d \ln r} - \frac{d \ln \rho}{d \ln r} \quad U = \frac{d \ln M_r}{d \ln r} \quad c_1 = \frac{r^3}{R_*^3} \frac{M_*}{M_r} \quad \Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_S$$

Thermal

$$\begin{aligned} \nabla &= \frac{d \ln T}{d \ln p} \quad \nabla_{\text{ad}} = \left(\frac{\partial \ln T}{\partial \ln p} \right)_S \quad \delta = - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \\ c_{\text{rad}} &= x^{-3} \frac{L_{\text{rad}}}{L_*} \quad c'_{\text{rad}} = \frac{d \ln c_{\text{rad}}}{d \ln r} \\ c_{\epsilon, \text{ad}} &= x^{-3} \frac{4\pi r^3 \epsilon_{\text{ad}} \rho}{L_*} \quad c_{\epsilon, S} = x^{-3} \frac{4\pi r^3 \epsilon_S \rho}{L_*} \quad c_{\text{thm}} = x^{-3} \frac{4\pi r^3 c_p T \rho}{L_*} \sqrt{\frac{GM_*}{R_*^3}} \\ c_{\text{kap}} &= (\kappa_{\text{ad}} - 4\nabla_{\text{ad}}) V \nabla + \nabla_{\text{ad}} \left(\frac{d \ln \nabla_{\text{ad}}}{d \ln r} + V \right) \\ \kappa_{\text{ad}} &= \left(\frac{\partial \ln \kappa}{\partial \ln p} \right)_S \quad \kappa_S = c_p T \left(\frac{\partial \ln \kappa}{\partial S} \right)_p \\ \epsilon_{\text{ad}} &= \left(\frac{\partial \epsilon}{\partial \ln p} \right)_S \quad \epsilon_S = c_p T \left(\frac{\partial \epsilon}{\partial S} \right)_p \end{aligned}$$